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Semi Modified Alpha Power Rayleigh Distribution: Its Properties and Applications

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ABSTRACT

A new probability distribution is developed in this study by adding an extra parameter to the existing Rayleigh distribution. The given study employed Rayleigh distribution as a baseline to the new probability generator called Semi Modified Alpha Power Rayleigh Distribution (SMAPRD). Several important statistical properties were developed for the new distribution such as, SF, HF, median, mode, order statistics, R^{th} moments, Mean Residual Life Function (MRLF) and entropy etc. Maximum likelihood estimation method was used to derive the estimates of the parameters. Two real data sets were applied to the proposed distribution and have a better fit as compare to the class of other distributions.

Introduction

Probability distribution is becoming a normal practice for researchers to improve and explore to new generation, while linking with modern technologies. Real life problems their analysis and complex data sets need the probability distributions accordingly from simplifying the classical distributions [1]. for achieving the purpose, we create new generators by adding some new parameters to the baseline distribution or merge the existing ones [2] and [3] Substituting a new parameter to the existing distribution. [4] proposed the T-X family of continuous distributions. [5] suggested beta generated distributions with beta as a parent distribution and cumulative distribution function (CDF) as a baseline of a continuous random variable. Later on, the beta transformation got replaced [6] with Kumaraswamy distribution. Univariate continuous distribution was constructed and reviewed generously by [7] for comparison purpose. A novel approach recently proposed by [8] called the alpha power transformation (APT) aiming the skewness into the baseline distribution by adding a new parameter in a continuous distribution. The Rayleigh distribution, a specific instance of the Weibull distribution, was first described by Lord Rayleigh in 1880 [9]. It is highly effective for modeling skewed data and is widely applied in disciplines such as oceanography, wireless communications, and signal processing. Its significance arises from its ability to represent the magnitude of a two-dimensional vector with independent, normally distributed components. The Rayleigh distribution has been employed in various fields. For instance, [10] Hoffman and Karst applied it to evaluate marine vessel performance, while [11] Dyer and Whisenand (1969) demonstrated its relevance in engineering applications. In medical research, [12] Bhattacharya and Tyagi utilized it for analyzing clinical data. Additionally, [13] Polovko highlighted its use in studying electro vacuum devices. Consequently, the distribution is a valuable tool for professionals in engineering, physics, and health sciences for modeling lifetime data.

Semi Modified Alpha Power Technique

The PDF and CDF of the Semi modified Alpha Power Technique are given by:

$$F(x) = \frac{F(x)(1-\alpha^{F(x)})}{(1-\alpha)} \qquad x > 0, \alpha > 0 \qquad (1)$$

$$f(x) = \frac{f(x)[\alpha^{F(x)} \log(\alpha) F(x) - (1-\alpha^{F(x)})]}{(\alpha-1)} \qquad x > 0, \alpha > 0 \qquad (2)$$

The F(x) and f(x) represents CDF and PDF of the baseline distribution.

This technique represents a good fit and shows efficient results in the class of other distributions [14], which are mentioned in the subsequent section with Rayleigh distribution as a baseline.

Semi Modified Alpha Power Rayleigh Distribution, its Statistical Properties and applications

The proposed probability generator specified in equation (1) is applied to Rayleigh distribution and a new distribution known as Semi Modified Alpha Power Rayleigh (SMAPRD) distribution is obtained.

CDF and PDF of Rayliegh distribution are as follows:

$$F(x) = 1 - e^{-(\lambda x)^2}; \quad \lambda, x \geq 0 \qquad (3)$$

$$f(x) = 2\lambda^2 x e^{-(\lambda x)^2} \qquad (4)$$

By using the CDF and PDF of Rayleigh Distribution as a baseline to the Semi modified alpha power technique will form as follows:

$$F(x) = \frac{(1-e^{-(\lambda x)^2})(1-\alpha^{1-e^{-(\lambda x)^2}})}{(1-\alpha)} \quad ; \alpha, \lambda, x \geq 0 \qquad (5)$$

$$f(x) = \frac{2\lambda^2 x e^{-(\lambda x)^2}[\alpha^{1-e^{-(\lambda x)^2}} \log(\alpha)\left(1-e^{-(\lambda x)^2}\right) - (1-\alpha^{(1-e^{-(\lambda x)^2})})]}{(\alpha-1)} \; ; \; \alpha, \lambda, x \geq 0 \qquad (6)$$

Eq (5) and eq(6) represents the CDF and PDF of SMAPR Distribution.

Graphical representation for SMAPRD of CDF and PDF with different parameter values:

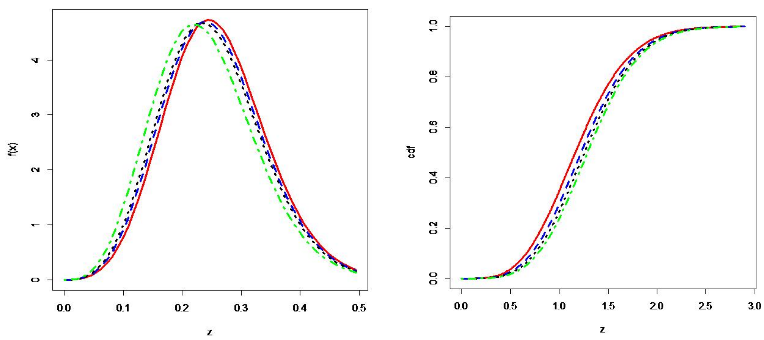

Figure 1: CDF and PDF of SMAPR Distribution.

Statistical Properties of SMAPR Distribution

Every distribution should have to satisfy some properties, among which some for SMAPR Distribution are given in this section below.

Survival Function of SMAPR Distribution

The Survival function $S_{SMAPRD}(x; \alpha, \lambda)$ is defined as, 1- CDF, by substituting eq (5),we get:

$$S(x) = 1 - F(x) = 1 - \frac{(1 - e^{-(\lambda x)^2})(1 - \alpha^{1-e^{-(\lambda x)^2}})}{(1 - \alpha)}$$

After simplification, the S.F becomes:

$$\boldsymbol{S(x)} = \frac{\alpha^{1-e^{-(\lambda x)^2}}\left[1-e^{-(\lambda x)^2}\right]+\boldsymbol{\alpha}-e^{-(\lambda x)^2}}{(\boldsymbol{\alpha}-\mathbf{1})} \qquad (7)$$

graphical representation of survival function for different parameter values carried out with different colors are given below.

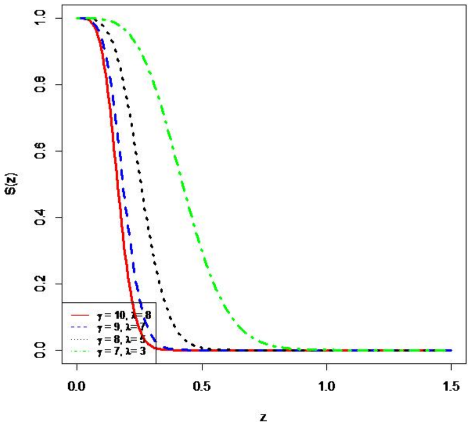

Figure 2: Survival Function of SMAPR Distribution.

Hazard Function of SMAPR Distribution

The hazard function $H_{SMAPRD}(x;\alpha,\lambda)$ is the ratio of PDF to its Survival function (SF) that has the following mathematical expression:

$$H(x) = \frac{pdf}{survival\,function} = \frac{f(x)}{s(x)} \qquad (8)$$

By putting eq(6) and eq(7) in eq(8) the Hazard function of SMAPR Distribution becomes:

$$H(x) = \frac{\dfrac{2\lambda^2 x e^{-(\lambda x)^2}[\alpha^{1-e^{-(\lambda x)^2}}\log(\alpha)\left(1-e^{-(\lambda x)^2}\right)-(1-\alpha^{(1-e^{-(\lambda x)^2})})]}{(\alpha-1)}}{\dfrac{\alpha^{1-e^{-(\lambda x)^2}}\left[1-e^{-(\lambda x)^2}\right]+\boldsymbol{\alpha}-e^{-(\lambda x)^2}}{(\boldsymbol{\alpha}-\mathbf{1})}}$$

After simplification, the hazard function becomes;

$$H(x) = \frac{2\lambda^2 x e^{-(\lambda x)^2}[\alpha^{1-e^{-(\lambda x)^2}}\log(\alpha)\left(1-e^{-(\lambda x)^2}\right)-(1-\alpha^{(1-e^{-(\lambda x)^2})})]}{\alpha^{1-e^{-(\lambda x)^2}}\left[1-e^{-(\lambda x)^2}\right]+\boldsymbol{\alpha}-e^{-(\lambda x)^2}} \qquad (9)$$

Quantile Function

Quantile function is defined as an inverse of the Distribution function.

$$F(X) = U$$
$$X = F^{-1}(U)$$

Where “U” follows the standard uniform distribution.

$$\boldsymbol{F(x)} = \frac{(1-e^{-(\lambda x)^2})(1-\alpha^{1-e^{-(\lambda x)^2}})}{(1-\alpha)} = \boldsymbol{U}$$

Shifting denominator to L.H.S.

$$\boldsymbol{F(x)} = (1-e^{-(\lambda x)^2})(1-\alpha^{1-e^{-(\lambda x)^2}}) = \boldsymbol{u}\,(\mathbf{1}-\boldsymbol{\alpha}) \qquad (10)$$

After simplification eq (6.12) becomes,

$$\boldsymbol{X} = [\boldsymbol{2log\lambda} - \boldsymbol{\lambda^2} - \boldsymbol{log}(\boldsymbol{log}(\boldsymbol{u}(\boldsymbol{1} - \boldsymbol{\alpha})))]^{\boldsymbol{1/2}} \qquad (11)$$

Is the required quantile function of SMAPR Distribution.

Median of SMAPR Distribution

Median can be obtained by putting u=1/2 in eq(11):

$$Median = \left[\boldsymbol{2log\lambda} - \boldsymbol{\lambda^2} - \boldsymbol{log}(\boldsymbol{log}(\frac{\boldsymbol{(1-\alpha)}}{\boldsymbol{2}}))\right]^{\boldsymbol{1/2}} \qquad (12)$$

Mode of SMAPR Distribution

Mode of the distribution is derived by taking derivative of eq (6):

$$\frac{\boldsymbol{d}}{\boldsymbol{dx}}\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$$

$$\frac{\boldsymbol{d}}{\boldsymbol{dx}}\boldsymbol{f}(\boldsymbol{x}) = \frac{\boldsymbol{d}}{\boldsymbol{dx}}\left[\frac{2\lambda^2 x e^{-(\lambda x)^2}[\alpha^{1-e^{-(\lambda x)^2}}\log(\alpha)\left(1-e^{-(\lambda x)^2}\right) - (1-\alpha^{(1-e^{-(\lambda x)^2})})]}{(\alpha-1)}\right] = 0$$

Taking out constant terms in coefficient and multiplied to L.H.S, it will become.

$$\frac{\boldsymbol{d}}{\boldsymbol{dx}}\boldsymbol{f}(\boldsymbol{x}) = \frac{\boldsymbol{d}}{\boldsymbol{dx}}\left[xe^{-(\lambda x)^2}[\alpha^{1-e^{-(\lambda x)^2}}\log(\alpha)\left(1-e^{-(\lambda x)^2}\right) - (1-\alpha^{(1-e^{-(\lambda x)^2})})]\right] = 0$$

Applying derivatives;

$$\frac{\boldsymbol{d}}{\boldsymbol{dx}}\boldsymbol{f}(\boldsymbol{x}) = \frac{\boldsymbol{d}}{\boldsymbol{dx}}\left[xe^{-(\lambda x)^2}\alpha^{1-e^{-(\lambda x)^2}}\log(\alpha)\left(1-e^{-(\lambda x)^2}\right)\right] - \frac{\boldsymbol{d}}{\boldsymbol{dx}}\left[xe^{-(\lambda x)^2}(1-\alpha^{(1-e^{-(\lambda x)^2})})\right] = 0$$

$$\frac{\boldsymbol{d}}{\boldsymbol{dx}}\boldsymbol{f}(\boldsymbol{x}) = \left[\begin{matrix}-\alpha\log(\alpha)e^{-3(\lambda x)^2}((2(\lambda x)^2-1)e^{2(\lambda x)^2} + ((-2log(\alpha)-4)(\lambda x)^2+1)e^{(\lambda x)^2} \\ +2\log(\alpha)(\lambda x)^2 + 2\log(\alpha)(\lambda x)^2))\end{matrix}\right] + \left[\frac{e^{-2(\lambda x)^2}\left[(\left(\alpha^{e^{-(\lambda x)^2}}-\alpha\right)(2(\lambda x)^2-1)e^{(\lambda x)^2} + 2\alpha\log(\alpha)(\lambda x)^2)\right]}{\alpha^{e^{-(\lambda x)^2}}}\right]$$

After simplifications the mode becomes;

$$\boldsymbol{Mode} = \left[\frac{\left[e^{-2(\lambda x)^2}\left(\begin{matrix}(\left(\alpha^{e^{-(\lambda x)^2}}-\alpha\right)(2(\lambda x)^2-1)e^{(\lambda x)^2}+2\alpha\log(\alpha)(\lambda x)^2) \\ -\alpha\log(\alpha)e^{-(\lambda x)^2}((2(\lambda x)^2-1)e^{2(\lambda x)^2}+((-2log(\alpha)-4)(\lambda x)^2+1)e^{(\lambda x)^2} \\ +2\log(\alpha)(\lambda x)^2)\end{matrix}\right)\right]}{\alpha^{e^{-(\lambda x)^2}}}\right];$$

$$\boldsymbol{\alpha}, \boldsymbol{\beta} > \boldsymbol{0} \quad \textbf{(13)}$$

eq (13) represents the final expression of Mode for SMAPR Distribution.

R$^{\text{th}}$ **Raw Moment**

$$\boldsymbol{\acute{\mu}_r} = \boldsymbol{E}(\boldsymbol{x})^{\boldsymbol{r}} = \int_{\boldsymbol{0}}^{\infty}\boldsymbol{x}^{\boldsymbol{r}}\,\boldsymbol{f}(\boldsymbol{x})\boldsymbol{dx}$$

$$\acute{\mu}_r = \int_0^\infty x^r \left[\frac{2\lambda^2 x e^{-(\lambda x)^2}[\alpha^{1-e^{-(\lambda x)^2}} \log(\alpha)\left(1-e^{-(\lambda x)^2}\right) - (1-\alpha^{(1-e^{-(\lambda x)^2})})]}{(\alpha-1)}\right] dx$$

Taking out constant terms in coefficient, we get

$$\acute{\mu}_r = \frac{2\lambda^2}{(\alpha-1)}\int_0^\infty \left[x^{r+1} e^{-(\lambda x)^2}[\alpha^{1-e^{-(\lambda x)^2}} \log(\alpha)\left(1-e^{-(\lambda x)^2}\right) - (1 - \alpha^{(1-e^{-(\lambda x)^2})})]\right] dx$$

Taking suppositions $y = 1 - e^{-(\lambda x)^2}$, $1 - y = e^{-(\lambda x)^2}$, $\frac{-[\log(1-y)]^{\frac{1}{2}}}{\lambda} = x$,

$$dx = \frac{dy}{2\lambda(1-y)(log(1-y))^{\frac{1}{2}}}$$

Limits if $x = 0$ then $y = 0$ & if $x = \infty$ then $y = 1$

$$\acute{\mu}_r = \frac{2\lambda^2}{(\alpha-1)}\int_0^1 \left[\left[\frac{-[\log(1-y)]^{\frac{1}{2}}}{\lambda}\right]^{r+1} (1-y)[\alpha^y \log(\alpha)\, y - (1 - \alpha^y)]\right] \frac{dy}{2\lambda(1-y)(log(1-y))^{\frac{1}{2}}}$$

After some cancelations and taking out constants, we get;

$$\acute{\mu}_r = \frac{-1}{\lambda^r(\alpha-1)}\int_0^\infty \left[[\log(1-y)]^{\frac{r}{2}}[\alpha^y \log(\alpha)\, y - (1-\alpha^y)]\right] dy$$

After taking integrals, the equation becomes.

$$\acute{\mu}_r = \frac{-1}{\lambda^r(\alpha-1)}\left[\Gamma\left(\frac{r+2}{2}\right)\left[\frac{\alpha(\log(\alpha)-1)+1}{\log(\alpha)} - \frac{\log(\alpha)-\alpha+1}{\log(\alpha)}\right]\right]$$

After simplifications, we get.

$$\acute{\mu}_r = \frac{-1}{\lambda^r(\alpha-1)}\Gamma\left(\frac{r+2}{2}\right)\left[\frac{log(\alpha)(\alpha-1)}{\log(\alpha)}\right]$$

Eq (5.16) is the required Rth moment:

$$\acute{\mu}_r = -\frac{\Gamma\left(\frac{r+2}{2}\right)}{\lambda^r} \qquad \textbf{(14)}$$

Moment Generating Function (MGF)

Moment generating function of SMAPR Distribution

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x)dx$$

Using exponent series, the equation becomes

$$M_x(t) = \int_0^\infty (1 + \frac{t^1x^1}{1!} + \frac{t^2x^2}{2!} + \ldots.)f(x)dx$$

$$M_x(t) = \int_0^\infty (1 + \frac{t^1x^1}{1!} + \frac{t^2x^2}{2!} + \ldots.)f(x)dx$$

$$M_x(t) = \int_0^\infty \sum_{r=0}^{\infty} \frac{t^r x^r}{r!} f(x)dx$$

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} x^r f(x) dx$$

Using Rth moments results, the mgf becomes:

$$M_x(t) =- \sum_{r=0}^{\infty} \frac{t^r \Gamma\left(\frac{r+2}{2}\right)}{r! \lambda^r} \qquad \textbf{(15)}$$

Eq(15) is the required MGF for SMAPR Distribution.

Order Statistics

Let $X_1, X_2, X_3, \ldots, X_n$ be the ordered random variables corresponding to a sample of size "n". the PDF of ith order statistics of SMAPR Distribution , is $f_{(i,n)}(X)$ given by the following expression

$$\boldsymbol{f_{(i,n)}(X) = \frac{n!}{(i-1)!(n-1)!} f(x) F(X)^{(i-1)} [1 - F(X)]^{(n-i)}} \qquad \textbf{(16)}$$

By substituting eq (6.7) and eq (6.8) in eq (6.18) ith order statistics , we have

$$\mathbf{f_{(i,n)}(X)} = \frac{n!}{(i-1)!\,(n-1)!}$$

$$\frac{2\lambda^2 x e^{-(\lambda x)^2} [\alpha^{1-e^{-(\lambda x)^2}} \log(\alpha) \left(1 - e^{-(\lambda x)^2}\right) - (1 - \alpha^{(1-e^{-(\lambda x)^2})})]}{(\alpha - 1)} \left[\frac{(1 - e^{-(\lambda x)^2})(1 - \alpha^{1-e^{-(\lambda x)^2}})}{(1-\alpha)}\right]^{(i-1)}$$

$$[1 - \frac{(1-e^{-(\lambda x)^2})(1-\alpha^{1-e^{-(\lambda x)^2}})}{(1-\alpha)}]^{(n-i)} \qquad \textbf{(17)}$$

Put $i = 1$ in (17) to have expression of smallest order statistic.

$$\mathbf{f_{(1,n)}(X)} = \frac{n!}{(1-1)!\,(n-1)!} \frac{2\lambda^2 x e^{-(\lambda x)^2} [\alpha^{1-e^{-(\lambda x)^2}} \log(\alpha) \left(1 - e^{-(\lambda x)^2}\right) - (1 - \alpha^{(1-e^{-(\lambda x)^2})})]}{(\alpha - 1)}$$

$$\left[\frac{(1 - e^{-(\lambda x)^2})(1 - \alpha^{1-e^{-(\lambda x)^2}})}{(1-\alpha)}\right]^{(1-1)} [1 - \frac{(1 - e^{-(\lambda x)^2})(1 - \alpha^{1-e^{-(\lambda x)^2}})}{(1-\alpha)}]^{(n-1)}$$

$$\mathbf{f_{(1,n)}(X)} = \frac{n!}{0!\,(n-1)!} \frac{2\lambda^2 x e^{-(\lambda x)^2} [\alpha^{1-e^{-(\lambda x)^2}} \log(\alpha) \left(1 - e^{-(\lambda x)^2}\right) - (1 - \alpha^{(1-e^{-(\lambda x)^2})})]}{(\alpha - 1)}$$

$$\left[\frac{(e^{-x^{-\beta}})(1 - \alpha^{e^{-x^{-\beta}}})}{(1-\alpha)}\right]^{(0)} [1 - \frac{(e^{-x^{-\beta}})(1 - \alpha^{e^{-x^{-\beta}}})}{(1-\alpha)}]^{(n-1)}$$

After simplification it becomes.

$$\mathbf{f_{(1,n)}(X)} = \frac{n!}{0!\,(n-1)!} \frac{2\lambda^2 x e^{-(\lambda x)^2} [\alpha^{1-e^{-(\lambda x)^2}} \log(\alpha) \left(1 - e^{-(\lambda x)^2}\right) - (1 - \alpha^{(1-e^{-(\lambda x)^2})})]}{(\alpha - 1)}$$

$$[1 - \frac{(1-e^{-(\lambda x)^2})(1-\alpha^{1-e^{-(\lambda x)^2}})}{(1-\alpha)}]^{(n-1)}$$

(18)

Put $i = n$ in (17), we get largest order statistic as follows:

$$f_{(n,n)}(X) = \frac{n!}{(n-1)!\,(n-1)!}$$

$$\frac{2\lambda^2 x e^{-(\lambda x)^2}[\alpha^{1-e^{-(\lambda x)^2}} \log(\alpha)\left(1-e^{-(\lambda x)^2}\right)-(1-\alpha^{(1-e^{-(\lambda x)^2})})]}{(\alpha-1)}\left[\frac{(1-e^{-(\lambda x)^2})(1-\alpha^{1-e^{-(\lambda x)^2}})}{(1-\alpha)}\right]^{(\boldsymbol{n-1})}$$

$$[1-\frac{(1-e^{-(\lambda x)^2})(1-\alpha^{1-e^{-(\lambda x)^2}})}{(1-\alpha)}]^{(\boldsymbol{n-n})}$$

The required largest order statistics is:

$$\mathrm{f}_{(\mathrm{n,n})}(\mathrm{X})=\frac{\mathrm{n}!}{(\mathrm{n}-1)!\,(\mathrm{n}-1)!}$$

$$\frac{2\lambda^2 x e^{-(\lambda x)^2}[\alpha^{1-e^{-(\lambda x)^2}} \log(\alpha)\left(1-e^{-(\lambda x)^2}\right)-(1-\alpha^{(1-e^{-(\lambda x)^2})})]}{(\alpha-1)}\left[\frac{(1-e^{-(\lambda x)^2})(1-\alpha^{1-e^{-(\lambda x)^2}})}{(1-\alpha)}\right]^{(\boldsymbol{n-1})}$$

(19)

Stress Strength Parameter

Suppose X and Y be two continuous and independent random variables, where $X \sim SMAPRD\ (\alpha_1, \lambda\ )$ and $Y \sim SMAPRD\ (\alpha_2, \lambda\ )$ then the stress strength parameter, say R, is defined as

$R = P(Y < X)$

Stress-strength reliability is:

$$R = P(Y < X) = \int_{-\infty}^{\infty} f_1(x) F_2(x) dx \qquad (20)$$

That is, the probability that **strength X exceed stress Y.**

$$R=\int_{-\infty}^{\infty} f_1(x;\alpha_1,\lambda).\,F_2(x;\alpha_2,\lambda)dx$$

Utilizing eq(6.7) and eq(6.8) in eq(6.22), we get;

$$R=\int_0^{\infty}\left[\frac{2\lambda^2 x e^{-(\lambda x)^2}[{\alpha_1}^{1-e^{-(\lambda x)^2}} \log(\alpha_1)\left(1-e^{-(\lambda x)^2}\right)-(1-{\alpha_1}^{(1-e^{-(\lambda x)^2})})]}{(\alpha_1-1)}\right]$$

$$\left[\frac{(1-e^{-(\lambda x)^2})(1-{\alpha_2}^{1-e^{-(\lambda x)^2}})}{(1-\alpha_2)}\right]\boldsymbol{dx}$$

Taking out constants in coefficient;

$$R=\frac{2\lambda^2}{(\alpha_1-1)(\boldsymbol{1-\alpha_2})}\int_0^{\infty}\left[x e^{-(\lambda x)^2}\left[{\alpha_1}^{1-e^{-(\lambda x)^2}}\log(\alpha_1)\left(1-e^{-(\lambda x)^2}\right)\right.\right.$$
$$\left.\left.-\left(1-{\alpha_1}^{\left(1-e^{-(\lambda x)^2}\right)}\right)\right]\right]\left[(1-e^{-(\lambda x)^2})(1-{\alpha_2}^{1-e^{-(\lambda x)^2}})\right]\boldsymbol{dx}$$

Taking suppositions $\mathrm{y}=1-e^{-(\lambda x)^2}$, $1-\mathrm{y}=e^{-(\lambda x)^2}$, $\frac{-[\log(1-y)]^{\frac{1}{2}}}{\lambda}=\mathrm{x}$,

$$\mathrm{dx}=\frac{\mathrm{dy}}{2\lambda(1-y)(log(1-y))^{\frac{1}{2}}}$$

Limits if $\boldsymbol{x=0}$ $\boldsymbol{then\ y=0}$ & $\boldsymbol{if\ x=\infty\ then\ y=1}$

$$R = \frac{2\lambda^2}{(\alpha_1 - 1)(\mathbf{1} - \boldsymbol{\alpha_2})} \int_0^1 \left[\frac{-[\log(1-y)]^{\frac{1}{2}}}{\lambda} (1-\mathrm{y})[\alpha_1{}^y \log(\alpha_1) y - (1 - \alpha_1{}^y)][y(1 - \alpha_2{}^y)] \right] \frac{\mathrm{dy}}{2\lambda(1-y)(log(1-y))^{\frac{1}{2}}}$$

After some cancelations and taking constants to the coefficients, we get;

$$R = \frac{-1}{(\alpha_1 - 1)(\mathbf{1} - \boldsymbol{\alpha_2})} \int_0^1 [[\alpha_1{}^y \log(\alpha_1) y - (1 - \alpha_1{}^y)][y(1 - \alpha_2{}^y)]] \,\mathrm{dy}$$

Multiplying terms with in the integrals, we get;

$$R = \frac{-1}{(\alpha_1 - 1)(\mathbf{1} - \boldsymbol{\alpha_2})} \int_0^1 \left[\begin{bmatrix} y\alpha_1{}^y (\mathrm{y}\log(\alpha_1) + 1) - y\alpha_1{}^y \alpha_2{}^y (\mathrm{y}\log(\alpha_1) + 1) \\ -y(1 - \alpha_2{}^y) \end{bmatrix} \right] \mathrm{dy}$$

$$R = \frac{-1}{(\alpha_1 - 1)(\mathbf{1} - \boldsymbol{\alpha_2})} \left[\left[\int_0^1 y\alpha_1{}^y (\mathrm{y}\log(\alpha_1) + 1) dy \right] - \left[\int_0^1 y\alpha_1{}^y \alpha_2{}^y (\mathrm{y}\log(\alpha_1) + 1) dy \right] - \left[\int_0^1 y(1 - \alpha_2{}^y) dy \right] \right]$$

by applying integrals, we get

$$R = \frac{-1}{(\alpha_1 - 1)(\mathbf{1} - \boldsymbol{\alpha_2})} \left[\left[\frac{\alpha_1 \log(\alpha_1)^2 - \alpha_1 \log(\alpha_1) + \alpha_1 - 1}{\log(\alpha_1)} \right] - \left[\frac{(\log(\alpha_1) + 1)(\alpha_1\alpha_2(\log(\alpha_2) + \log(\alpha_1) - 1) + 1)}{(\log(\alpha_2) + \log(\alpha_1))^2} \right] - \left[\frac{\log(\alpha_2)^2 - 2\alpha_2 \log(\alpha_2) + 2\alpha_2 - 2}{2\log(\alpha_2)^2} \right] \right]$$

After simplification, we get;

$$R = \frac{-1}{(\alpha_1 - 1)(\mathbf{1} - \boldsymbol{\alpha_2})} \left[\left[\frac{2\alpha_1 \log(\alpha_1) \log(\alpha_2)^2 \begin{bmatrix} log(\alpha_2)(\log(\alpha_1)(\log(\alpha_2) + 2\log(\alpha_1) - 2) \\ -(2 - \log(\alpha_2)) + \log(\alpha_1)(1 - \log(\alpha_1)^2 - \log(\alpha_1)) \end{bmatrix} - [2\log(\alpha_2)^2 (\log(\alpha_2)^2 (\alpha_1 - 1) - \log(\alpha_1)(2\log(\alpha_2) + \log(\alpha_1)))]}{\log(\alpha_1)(\log(\alpha_2) + \log(\alpha_1))^2 2\log(\alpha_2)^2} \right] \right] \quad (21)$$

This is the required results in eq(21) for stress strength of SMAPR Distribution.

Renyi Entropy (RE)

Let X ~ *SMAPRD (α,λ), then the renyi entropy result is given as:*

$$R.E_X = -\frac{p}{1-p} log\left(\frac{2\lambda^2}{(\boldsymbol{\alpha} - \mathbf{1})}\right)^{p-1}$$

Prof: by definition

$$S.E_X = \frac{1}{1-p} log\left[\int_{-\infty}^{+\infty} f(x)^p dx\right]$$

Putting eq(8), we get:

$R.E_X$

$$= \frac{1}{1-p} log\left[\int_{-\infty}^{+\infty}\left[\frac{2\lambda^2 x e^{-(\lambda x)^2}[\alpha^{1-e^{-(\lambda x)^2}} \log(\alpha)\left(1-e^{-(\lambda x)^2}\right)-(1-\alpha^{(1-e^{-(\lambda x)^2})})]}{(\alpha-1)}\right]^p dx\right]$$

Taking constants to the coefficient and applying log power rule:

$$R\ .E_X = \frac{1}{1-p} log\left(\frac{2\lambda^2}{(\boldsymbol{\alpha}-\mathbf{1})}\right)^p \left[p.\,log \int_0^{\infty}\left(x e^{-(\lambda x)^2}[\alpha^{1-e^{-(\lambda x)^2}} \log(\alpha)\left(1-e^{-(\lambda x)^2}\right)-(1-\alpha^{(1-e^{-(\lambda x)^2})})\right)\right] dx$$

Taking suppositions $y = 1-e^{-(\lambda x)^2}$, $1-y = e^{-(\lambda x)^2}$, $\frac{-[\log(1-y)]^{\frac{1}{2}}}{\lambda} = x$,

$$dx = \frac{dy}{2\lambda(1-y)(log(1-y))^{\frac{1}{2}}}$$

Limits if $x = 0$ *then* $y = 0$ & *if* $x = \infty$ *then* $y = 1$

$$R.E_X = \frac{1}{1-p} log\left(\frac{2\lambda^2}{(\boldsymbol{\alpha}-\mathbf{1})}\right)^p \left[p.\,log\int_0^1\left(\frac{-[\log(1-y)]^{\frac{1}{2}}}{\lambda}(1-y)[\alpha^y \log(\alpha)\, y - (1-\alpha^y))]\frac{dy}{2\lambda(1-y)(log(1-y))^{\frac{1}{2}}}\right.\right.$$

Due to some cancelations and taking out constants in coefficient, we get;

$$R.E_X = \frac{p}{1-p} log\left(\frac{2\lambda^2}{(\boldsymbol{\alpha}-\mathbf{1})}\right)^p log\left(\frac{1}{2\lambda^2}\right)\left[-log\int_0^1([\alpha^y \log(\alpha)\, y-(1-\alpha^y))\right] dy$$

Taking integrals:

$$R.E_X = -\frac{p}{1-p} log\left(\frac{2\lambda^2}{(\boldsymbol{\alpha}-\mathbf{1})}\right)^p log\left(\frac{1}{2\lambda^2}\right)\left[log\left[\frac{\alpha(\log(\alpha)-1)+1}{\log(\alpha)} - \frac{\log(\alpha)-\alpha+1}{\log(\alpha)}\right]\right]$$

After simplifications, we get:

$$R.E_X = -\frac{p}{1-p} log\left(\frac{2\lambda^2}{(\boldsymbol{\alpha}-\mathbf{1})}\right)^p log\left(\frac{(\boldsymbol{\alpha}-\mathbf{1})}{2\lambda^2}\right)$$

This is the required results of renyi entropy in eq (22)

$$R.E_X = -\frac{p}{1-p} log\left(\frac{2\lambda^2}{(\boldsymbol{\alpha}-\mathbf{1})}\right)^{p-1} \quad \textbf{(22)}$$

Mean Residual Life Function (MRLF)

The MRLF is the average remaining life of a component that has survived till time t. Here X is lifetime of an object with *f(x)* and *s(x)* given in eq (6) and eq(7) respectively.

$$\mu(t) = \frac{1}{S(t)}\left[E(t) - \int_0^t x f(x) dx\right] \quad (23)$$

Where

$$\int_0^t xf(x)dx = \int_0^t x\left(\frac{2\lambda^2 x e^{-(\lambda x)^2}[\alpha^{1-e^{-(\lambda x)^2}} \log(\alpha)\left(1-e^{-(\lambda x)^2}\right) - (1-\alpha^{(1-e^{-(\lambda x)^2})})]}{(\alpha-1)}\right)dx$$

Taking out constants in coefficient.

$$\int_0^t xf(x)dx = \frac{2\lambda^2}{(\alpha-1)}\int_0^t x^2 e^{-(\lambda x)^2}[\alpha^{1-e^{-(\lambda x)^2}} \log(\alpha)\left(1-e^{-(\lambda x)^2}\right) - (1 - \alpha^{(1-e^{-(\lambda x)^2})})]]\, dx$$

Taking suppositions $\mathrm{y} = 1 - e^{-(\lambda x)^2}$, $1 - \mathrm{y} = e^{-(\lambda x)^2}$, $\frac{-[\log(1-y)]^{\frac{1}{2}}}{\lambda} = \mathrm{x}$,

$$\mathrm{dx} = \frac{\mathrm{dy}}{2\lambda(1-y)(log(1-y))^{\frac{1}{2}}}$$

Limits if $x = 0$ then $y = 0$ & if $x = t$, then $y = 1 - e^{-(tx)^2}$

$$\int_0^t xf(x)dx = \frac{2\lambda^2}{(\alpha-1)}\int_0^{1-e^{-(tx)^2}}\left[\frac{-[\log(1-y)]^{\frac{1}{2}}}{\lambda}\right]^2 (1-\mathrm{y})[\alpha^y \log(\alpha)\, y - (1 - \alpha^y)]]\frac{\mathrm{dy}}{2\lambda(1-y)(log(1-y))^{\frac{1}{2}}}$$

$$\int_0^t xf(x)dx = \frac{-1}{\lambda(\alpha-1)}\int_0^{1-e^{-(tx)^2}} [\log(1-y)]^{\frac{1}{2}}[\alpha^y \log(\alpha)\, y - (1-\alpha^y)]]\,\mathrm{dy}$$

$$\int_0^t xf(x)dx = -\frac{1}{\lambda(\alpha-1)}\left[\int_0^{1-e^{-(tx)^2}}\left[\frac{\alpha^y \log(\alpha)\, y}{[\log(1-y)]^{\frac{1}{2}}}\right]dy - \int_0^{1-e^{-(tx)^2}}\left[\frac{1-\alpha^y}{[\log(1-y)]^{\frac{1}{2}}}\right]\mathrm{dy}\right]$$

By applying integral, we get

$$\int_0^t xf(x)dx = -\frac{1}{\lambda(\alpha-1)}\left[\left[-\frac{\sqrt{\pi}ief(itx) + 2txe^{(tx)^2}}{2}\right]\left[-\frac{\left(\left(\alpha\log(\alpha)e^{(tx)^2} - \alpha\log(\alpha)\right)e^{e^{-\log(\alpha)(tx)^2}} - 1\right)}{\log(\alpha)} + \frac{\alpha^{e^{(tx)^2}}\log(\alpha)\, e^{(tx)^2} - \alpha^{e^{(tx)^2}}(\log(\alpha)-1) - \alpha}{\alpha^{e^{(tx)^2}}\log(\alpha)}\right]\right]$$

After simplification

$$\int_0^t xf(x)dx =$$

$$\frac{1}{\lambda(\alpha-1)}\left[\left[\frac{\sqrt{\pi}ief(itx)+2txe^{(tx)^2}}{2}\right]\left[\frac{\alpha^{e^{(tx)^2}}\left(\log(\alpha)\,e^{(tx)^2}+(\log(\alpha)-1)-\left(\alpha\log(\alpha)e^{(tx)^2}-\alpha\log(\alpha)\right)e^{e^{-\log(\alpha)\,(tx)^2}}-\alpha-1\right)}{\alpha^{e^{(tx)^2}}\log(\alpha)}\right]\right]$$

$$\int_0^t xf(x)dx = \frac{1}{2\lambda\log(\alpha)\,(\alpha-1)}\left[\left[\sqrt{\pi}ief(itx)+\right.\right.$$

$$\left.2txe^{(tx)^2}\right]\left[\frac{\alpha^{e^{(tx)^2}}\left(\log(\alpha)\,e^{(tx)^2}+(\log(\alpha)-1)-\left(\alpha\log(\alpha)e^{(tx)^2}-\alpha\log(\alpha)\right)e^{e^{-\log(\alpha)\,(tx)^2}}-\alpha-1\right)}{\alpha^{e^{(tx)^2}}}\right]\Bigg]$$

(24)

E(t) from eq(23), we get.

$$\boldsymbol{E}(\boldsymbol{t}) =-\frac{\Gamma\left(\frac{r+2}{2}\right)}{\lambda^r} \qquad (25)$$

Pitting eq (6.9), eq(6.26) and eq(6.27) in eq(6.25),we get;

$$\mu(t)=\frac{(\alpha-1)}{\alpha^{1-e^{-(\lambda x)^2}}\left[1-e^{-(\lambda x)^2}\right]+\boldsymbol{\alpha}-e^{-(\lambda x)^2}}$$

$$\left[\begin{array}{c}\frac{\Gamma\left(\frac{r+2}{2}\right)}{\lambda^r}+\frac{\left[\sqrt{\pi}ief(itx)+2txe^{(tx)^2}\right]}{2\lambda\log(\alpha)\,(\alpha-1)}\\ \left[\left[\frac{\alpha^{e^{(tx)^2}}\left(\log(\alpha)\,e^{(tx)^2}+(\log(\alpha)-1)-\left(\alpha\log(\alpha)e^{(tx)^2}-\alpha\log(\alpha)\right)e^{e^{-\log(\alpha)\,(tx)^2}}-\alpha-1\right)}{\alpha^{e^{(tx)^2}}}\right]\right]\end{array}\right] \qquad (26)$$

Is the required MRLF in eq(26).

Applications

To verify the performance of SMAPR Distribution, two real data sets were used and comparing the results with different Rayleigh family of distributions by model selection criteria's to demonstrate the performance of the proposed SMAPR model.

Data set 1: The first data set is related with the monthly actual taxes revenue in Egypt from January 2006 to November 2010.The data has been analyzed by [15]. The data values are as follows.

5.9	20.4	14.9	16.2	17.2	7.8	6.1	9.2	10.2	9.6	13.3	8.5	21.6	18.5	5.1
6.7	17	8.6	9.7	39.2	35.7	15.7	9.7	10	4.1	36	8.5	8	26.2	21.9
16.7	21.3	35.4	14.3	8.5	10.6	19.1	20.5	7.1	7.7	18.1	16.5	11.9	7	8.6
12.5	10.3	11.2	6.1	8.4	11	11.6	11.9	5.2	6.8	8.9	7.1	10.8		

Data set 2: The second data set is selected from El-Bassiouny et al. [16], related to failure times of aircraft windshields. The data points are as follows:

3.70	2.74	2.73	2.50	3.60	3.11	3.27	2.87	1.47	3.11
4.42	2.41	3.19	3.22	1.69	3.28	3.09	1.87	3.15	4.90
3.75	2.43	2.95	2.97	3.39	2.96	2.53	2.67	2.93	3.22
3.39	2.81	4.20	3.33	2.55	3.31	3.31	2.85	2.56	3.56
3.15	2.35	2.55	2.59	2.38	2.81	2.77	2.17	2.83	1.92
1.41	3.68	2.97	1.36	0.98	2.76	4.91	3.68	1.84	1.59

3.19	1.57	0.81	5.56	1.73	1.59	2.00	1.22	1.12	1.71
2.17	1.17	5.08	2.48	1.18	3.51	2.17	1.69	1.25	4.38
1.84	0.39	3.68	2.48	0.85	1.61	2.79	4.70	2.03	1.80
1.57	1.08	2.03	1.61	2.12	1.89	2.88	2.82	2.05	3.65

The output of SMAPR Distribution compared with other Rayleigh distributions by standard model selection criteria. The results are shown in Table 2 and 3 for data set 1 and 2 respectively.

Table 2: MLEs and Model selection criteria results for data set 1

Distribution	**MLE**		**AIC**	**CAIC**	**BIC**	**HQIC**	**p-value**
SMAPRD	0.005537716 -	-0.065637032	388.6222	388.8404	392.7431	390.2274	0.03461
TPRD	0.004685541	1.435383720	392.5315	392.7497	396.6524	394.1367	0.01753
GRD	1.01654520	-0.06401146	393.5287	393.7468	397.6496	395.1338	0.07239
RD	0.06345339		391.542	391.6135	393.6025	392.3446	0.07315
LRD	0.4868529	21.4946692	433.2262	433.4444	437.3471	434.8314	1.148e-05
EIRD	3.4271287 0.9558138	3.4271287 0.9558138	454.4818	454.6055	459.6921	456.5905	0.002544

Table 3: MLEs and Model selection criteria results for data set 2

Distribution	**MLE**			**AIC**	**CAIC**	**BIC**	**HQIC**	**P-value**
SMAPRD	2.5656776	-0.4600956		288.6307	288.7544	293.8411	290.7394	0.8122
TPRD	0.1600898	0.3362184		292.9634	293.0871	298.1737	295.0721	0.1712
EIRD	3.4271287	0.9558138		354.4818	354.6055	359.6921	356.5905	0.00254
RD	0.3560144			301.0018	301.0426	303.607	302.0562	0.04356
LRD	4.41004	29.03294		314.7318	314.8555	319.9421	316.8405	0.00543
WRD	7.99205464	0.78982308	0.02156232	367.69671	369.19671	370.68391	368.27984	0.2423

In table 2 and 3 it is clearly obvious that the proposed distribution SMAPRD has less AIC,CAIC, BIC and HQIC than alternative fitted distributions. Therefore, it is concluded that the proposed distribution performance is better than other family of Rayleigh distributions.

Graphical representation of Q-Q and P-P plot for SMAPR Distribution of data set 1.

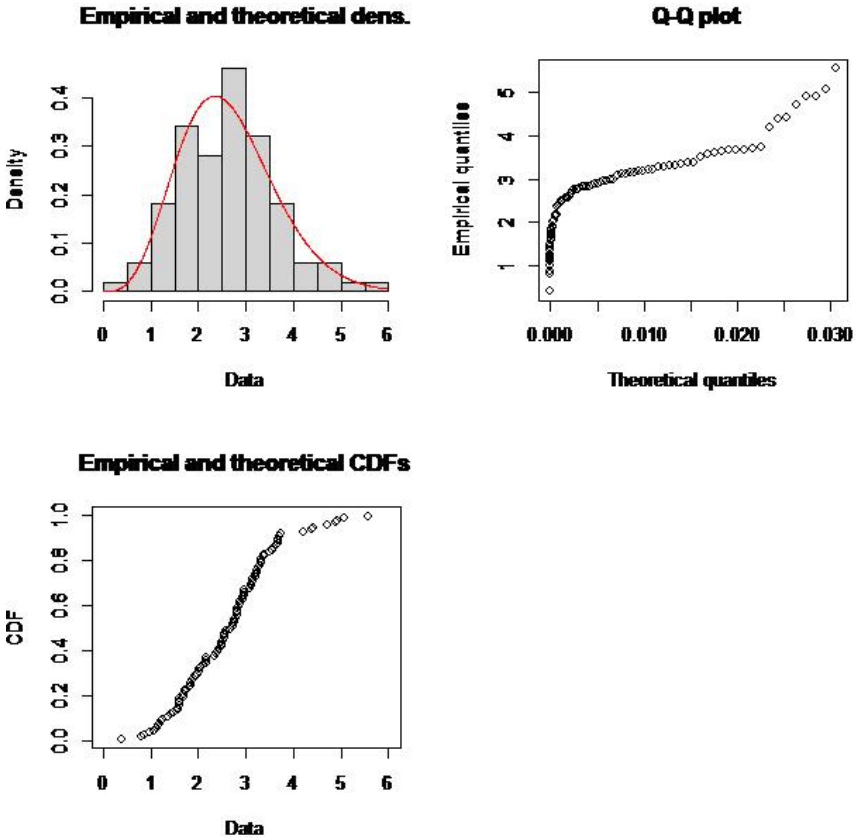

Figure 3: Q-Q and P-P plot of SMAPRD.

Graphical representation of Q-Q and P-P plot for SMAPR Distribution of data set 2.

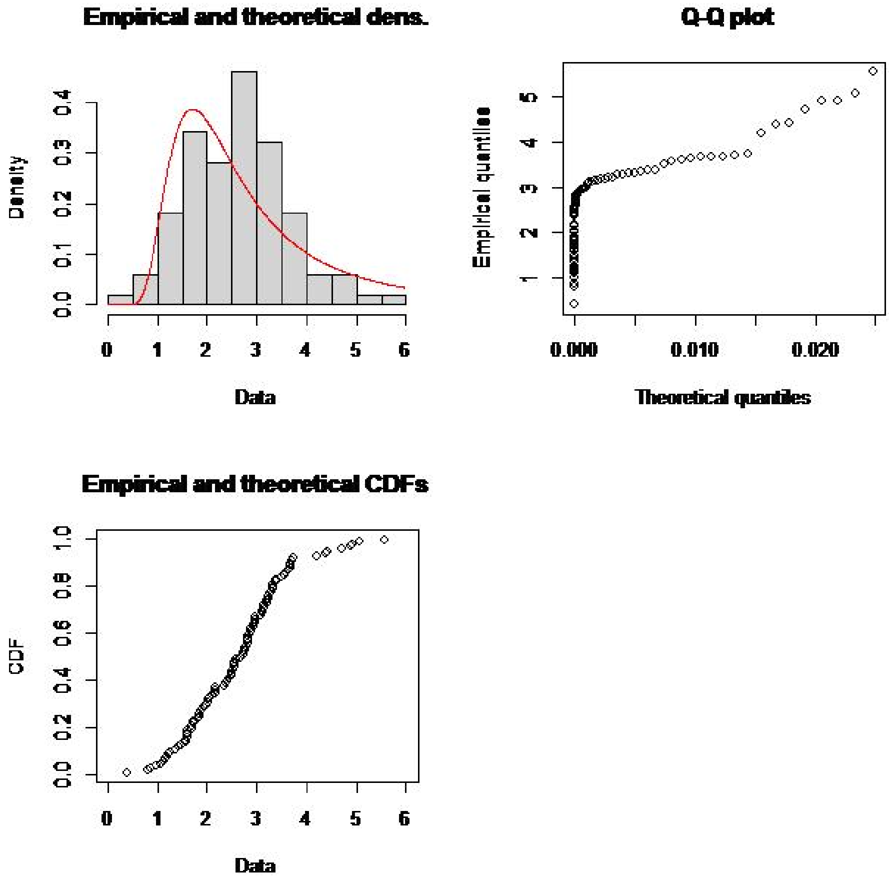

Figure 4: Q-Q and P-P plot of SMAPRD

Figure 4 and **5** shows the **Q-Q** and **P-P** plot of SMAPR Distribution shows the emparical and theoretical Densities which indicated that the proposed model provides a better fit for two real life data sets.

Conclusion

A new distribution has been developed as SMAPR Distribution using SMAP Technique. Several mathematical properties were discovered for the proposed distribution, i.e. Mgf, Qf, Median, Mode, Order statistics, Entropy measures, mean residual function etc. have been derived and MLE Technique was used for parameter estimation. For checking model performance, two real data sets were applied to the proposed model and it shows best results and better fit than the class of other Rayliegh distribution.

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